Eur. Phys. J. C 42, 475–481 (2005)
Digital Object Identifier (DOI) 10.1140/epjc/s2005-02344-4

THE EUROPEAN
PHYSICAL JOURNAL C

BRST symmetry of $SU(2)$ Yang–Mills theory in Cho–Faddeev–Niemi decomposition

K.-I. Kondo[1,2,a], T. Murakami[2,b], T. Shinohara[2,c]

[1] Department of Physics, Faculty of Science, Chiba University, Chiba 263-8522, Japan
[2] Graduate School of Science and Technology, Chiba University, Chiba 263-8522, Japan

Received: 26 April 2005 / Revised version: 27 May 2005 /
Published online: 19 July 2005 – © Springer-Verlag / Società Italiana di Fisica 2005

Abstract. We determine the nilpotent BRST and anti-BRST transformations for the Cho–Faddeev–Niemi variables for the $SU(2)$ Yang–Mills theory based on the new interpretation given in a previous paper of the Cho–Faddeev–Niemi decomposition. This gives a firm ground for performing the BRST quantization of the Yang–Mills theory written in terms of the Cho–Faddeev–Niemi variables. We propose also a modified version of the new maximal Abelian gauge which could play an important role in the reduction to the original Yang–Mills theory.

PACS. 12.38.Aw, 12.38.Lg

1 Introduction

The change of variables is known to be a useful method of extracting the most relevant degrees of freedom to the physics in question. A proposal along this direction for the topological degrees of freedom in Yang–Mills theory was first made by Cho [1], which has recently been readdressed by Faddeev and Niemi [2], and another one by Faddeev and Niemi [3]. The former is called the Cho–Faddeev–Niemi (CFN) decomposition in the literature, while the latter is called the Faddeev–Niemi decomposition. We focus on the former in this paper.

In applying the CFN decomposition, however, there was much controversy [4,5] over the treatment and the interpretation. A lot of progress toward the resolution of the conceptual issues was already made in [8,11], while the CFN decomposition was applied to reveal various non-perturbative features overlooked so far [3,5–8,11,12]. For example, the Skyrme–Faddeev model [6] describing the glueball as a knot soliton solution may be deduced from the Yang–Mills theory by way of the CFN decomposition [3,5,7,8,11,12]. Moreover, the instability of the Savvidy vacuum [9] disappears by eliminating a tachyon mode [10]; see [11] for the massless gluon and [12,13] for the massive gluon caused by a novel magnetic condensation.

In the previous paper [14], we have given a new interpretation of the Yang–Mills theory written in terms of the CFN variables, called the CFN–Yang–Mills theory. This interpretation enables us to elucidate how the CFN–Yang–Mills theory with the enlarged local gauge symmetry is reduced to the gauge theory with the same local and global gauge symmetries (Yang–Mills theory II) as the original Yang–Mills theory (Yang–Mills theory I). In fact, this is achieved by imposing a new version of the maximal Abelian gauge (new MAG), which plays a role quite different from the conventional MAG [15,16]. The new interpretation disposes of the pending question of the discrepancy between the CFN–Yang–Mills theory and the original Yang–Mills theory for independent degrees of freedom.

In the paper [13], we have already discussed how to perform the numerical simulations based on this interpretation and we have actually performed the Monte Carlo simulations on a lattice. This is the first implementation of the CFN decomposition on a lattice.

In this paper, moreover, we will discuss how to perform the BRST quantization of the CFN–Yang–Mills theory in the continuum formulation, as announced in [14]. For this purpose, we must first determine the nilpotent BRST transformations for the ghost–antighost fields and the Nakanishi–Lautrup auxiliary fields, in addition to those for the original CFN variables. Here it should be remarked that we must introduce more ghost and antighost fields reflecting the enlarged local gauge symmetry of the CFN–Yang–Mills theory. These BRST transformations are determined independently from the gauge fixing condition, i.e., the new MAG, as usual.

In order to fix the whole of the gauge degrees of freedom of the CFN–Yang–Mills theory, we must impose one more gauge-fixing condition, e.g., the Lorentz covariant gauge $\partial^\mu \mathrm{A}_\mu = 0$, in addition to the new MAG. The additional gauge fixing is necessary to fix the local gauge symmetry II in the Yang–Mills theory II which is obtained by taking the

[a] e-mail: kondok@faculty.chiba-u.jp
[b] e-mail: tom@cuphd.nd.chiba-u.ac.jp
[c] e-mail: sinohara@cuphd.nd.chiba-u.ac.jp

new MAG from the CFN–Yang–Mills theory. Then, we can obtain the explicit form of the Faddeev–Popov (FP) ghost terms associated to two gauge-fixing conditions, once the additional gauge-fixing condition is specified. Therefore, we can uniquely determine the total Lagrangian, based on which the BRST quantization is performed.

Moreover, we introduce another gauge transformation and the corresponding BRST transformation, called the primed transformations. This gives an alternative but equivalent description of the gauge and BRST symmetries of the CFN–Yang–Mills theory. Finally, we determine also the nilpotent anti-BRST transformation for the CFN variables. As an application, we propose a modified version of the new MAG, which is both BRST and anti-BRST exact simultaneously.

2 Yang–Mills theory in the CFN decomposition

2.1 Local gauge symmetry in terms of the CFN variables

The Cho–Faddeev–Niemi (CFN) decomposition (or change of variables) of the original Yang–Mills gauge field $\mathrm{A}_\mu(x)$ is performed as follows. We restrict our considerations to the gauge group $G = SU(2)$. First, we introduce a unit vector field $\boldsymbol{n}(x)$, i.e.,

$$\boldsymbol{n}(x)\cdot\boldsymbol{n}(x) := n^A(x)n^A(x) = 1 \quad (A=1,2,3). \tag{1}$$

Then the CFN decomposition is written in the form

$$\mathrm{A}_\mu(x) = c_\mu(x)\boldsymbol{n}(x) + g^{-1}\partial_\mu\boldsymbol{n}(x)\times\boldsymbol{n}(x) + \mathbb{X}_\mu(x), \tag{2}$$

where $\mathbb{X}_\mu(x)$ is perpendicular to $\boldsymbol{n}$,

$$\boldsymbol{n}(x)\cdot\mathbb{X}_\mu(x) = 0. \tag{3}$$

For later convenience, we introduce

$$\begin{aligned}\mathbb{V}_\mu(x) &= \mathbb{C}_\mu(x) + \mathbb{B}_\mu(x)\\ &= c_\mu(x)\boldsymbol{n}(x) + g^{-1}\partial_\mu\boldsymbol{n}(x)\times\boldsymbol{n}(x).\end{aligned} \tag{4}$$

The first important observation made in [14] is that the restricted potential c_μ and the gauge covariant potential $\mathbb{X}_\mu$ are specified by $\boldsymbol{n}$ and A_μ:

$$c_\mu(x) = \boldsymbol{n}(x)\cdot\mathrm{A}_\mu(x), \tag{5}$$

$$\mathbb{X}_\mu(x) = g^{-1}\boldsymbol{n}(x)\times D_\mu[\mathrm{A}]\boldsymbol{n}(x), \tag{6}$$

and, therefore, *the local gauge transformations* δc_μ *and* $\delta\mathbb{X}_\mu$ *are uniquely determined, once only the transformations* $\delta\boldsymbol{n}$ *and* $\delta\mathrm{A}_\mu$ *are specified.* This fact indicates the special role played by the $\boldsymbol{n}$ field in the gauge transformation.

In the previous paper [14], we have considered the local gauge symmetry respected by the Yang–Mills theory expressed in terms of the CFN variables, which we call *CFN–Yang–Mills theory* for short. We have shown [14] that *the CFN–Yang–Mills theory has the local gauge symmetry* $SU(2)^\omega_{\mathrm{local}}\times[SU(2)/U(1)]^\theta_{\mathrm{local}}$ *which is* larger *than the local* $SU(2)$ *symmetry of the original Yang–Mills theory*, since we can rotate the CFN variable $\boldsymbol{n}(x)$ by the angle $\boldsymbol{\theta}^\perp(x)$ independently of the gauge transformation parameter $\boldsymbol{\omega}(x)$ of $\mathrm{A}_\mu(x)$. Here the local gauge transformations of the CFN variables are given by

$$\delta\mathrm{A}_\mu(x) = D_\mu[\mathrm{A}]\boldsymbol{\omega}(x), \tag{7a}$$

$$\delta\boldsymbol{n}(x) = g\boldsymbol{n}(x)\times\boldsymbol{\theta}(x) = g\boldsymbol{n}(x)\times\boldsymbol{\theta}_\perp(x), \tag{7b}$$

$$\begin{aligned}\delta c_\mu(x) &= g(\boldsymbol{n}(x)\times\mathrm{A}_\mu(x))\cdot(\boldsymbol{\omega}_\perp(x)-\boldsymbol{\theta}_\perp(x))\\ &\quad+\boldsymbol{n}(x)\cdot\partial_\mu\boldsymbol{\omega}(x),\end{aligned} \tag{7c}$$

$$\begin{aligned}\delta\mathbb{X}_\mu(x) &= g\mathbb{X}_\mu(x)\times(\boldsymbol{\omega}_\parallel(x)+\boldsymbol{\theta}_\perp(x))\\ &\quad+D_\mu[\mathbb{V}](\boldsymbol{\omega}_\perp(x)-\boldsymbol{\theta}_\perp(x)),\end{aligned} \tag{7d}$$

where $_\parallel$ and $_\perp$ denote the parallel and perpendicular part to $\boldsymbol{n}$, and we have applied the gauge transformation (7a) and (7b) to (5) and (6) to obtain (7c) and (7d). If $\boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x)$, the transformations (7c) and (7d) reduce to gauge transformation II [12, 14] with the parameter $\boldsymbol{\omega}'(x) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x))$. Therefore, the gauge transformation II corresponds to the special case $\boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x)$.

Local gauge transformation II is

$$\delta'_\omega\boldsymbol{n} = g\boldsymbol{n}\times\boldsymbol{\omega}', \tag{8a}$$

$$\delta'_\omega c_\mu = \boldsymbol{n}\cdot\partial_\mu\boldsymbol{\omega}', \tag{8b}$$

$$\delta'_\omega\mathbb{X}_\mu = g\mathbb{X}_\mu\times\boldsymbol{\omega}', \tag{8c}$$

$$\Longrightarrow\quad \delta'_\omega\mathbb{V}_\mu = D_\mu[\mathbb{V}]\boldsymbol{\omega}'. \tag{8d}$$

This should be compared with[1] local gauge transformation I:

$$\delta_\omega\boldsymbol{n} = 0, \tag{9a}$$

$$\delta_\omega c_\mu = \boldsymbol{n}\cdot D_\mu[\mathrm{A}]\boldsymbol{\omega}, \tag{9b}$$

$$\delta_\omega\mathbb{X}_\mu = D_\mu[\mathrm{A}]\boldsymbol{\omega} - \boldsymbol{n}(\boldsymbol{n}\cdot D_\mu[\mathrm{A}]\boldsymbol{\omega}), \tag{9c}$$

$$\Longrightarrow\quad \delta_\omega\mathbb{V}_\mu = \boldsymbol{n}(\boldsymbol{n}\cdot D_\mu[\mathrm{A}]\boldsymbol{\omega}). \tag{9d}$$

2.2 A new interpretation of the CFN–Yang–Mills theory

In order to obtain the gauge theory with the same local gauge symmetry as the original Yang–Mills theory, therefore, we proceed by imposing a gauge-fixing condition by which $SU(2)^\omega_{\mathrm{local}}\times[SU(2)/U(1)]^\theta_{\mathrm{local}}$ is broken down to $SU(2)$ which is a subgroup of $SU(2)^\omega_{\mathrm{local}}\times[SU(2)/U(1)]^\theta_{\mathrm{local}}$. In the previous paper [14], we have found that a way to do this is to impose the minimizing condition

$$0 = \delta\int \mathrm{d}^D x\,\frac{1}{2}\mathbb{X}_\mu^2, \tag{10}$$

[1] Gauge transformation I was called the passive or quantum gauge transformation, while II was called the active or background gauge transformation. However, this classification is sometimes confusing and could lead to misleading results, since the two gauge transformations I and II are not independent.

which we called the new maximal Abelian gauge (new MAG). This is shown as follows. Since the relationship (6) leads to $g^2\mathbb{X}_\mu^2 = (D_\mu[\mathrm{A}]\boldsymbol{n})^2$, the local gauge transformation of $\mathbb{X}^2$ is calculated to be

$$\delta\frac{1}{2}\mathbb{X}_\mu^2 = g^{-1}(D_\mu[\mathrm{A}]\boldsymbol{n})\cdot\{D_\mu[\mathrm{A}](\boldsymbol{\omega}_\perp - \boldsymbol{\theta}_\perp)\times\boldsymbol{n}\}, \quad (11)$$

where we have used (7b) and (7a). Therefore, the local gauge transformation II does not change $\mathbb{X}^2$. Then the average over the spacetime of (11) reads

$$\delta\int \mathrm{d}^D x\frac{1}{2}\mathbb{X}_\mu^2 = -\int \mathrm{d}^D x(\boldsymbol{\omega}_\perp - \boldsymbol{\theta}_\perp)\cdot D_\mu[\mathbb{V}]\mathbb{X}_\mu, \quad (12)$$

where we have used (6) and integration by parts. Hence the minimizing condition (10) for arbitrary $\boldsymbol{\omega}_\perp$ and $\boldsymbol{\theta}_\perp$ yields a condition in the differential form

$$\mathbb{F}_{\mathrm{MA}} = \boldsymbol{\chi} := D_\mu[\mathbb{V}]\mathbb{X}_\mu \equiv 0. \quad (13)$$

Note that (13) denotes two conditions, since $\boldsymbol{n}\cdot\boldsymbol{\chi} = 0$ which follows from the identity $\boldsymbol{n}\cdot D_\mu[\mathbb{V}]\mathbb{X}_\mu = 0$. Therefore, the minimization condition (10) works as a gauge-fixing condition except for gauge transformation II, i.e., $\boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x)$. For $\boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x)$, the condition (13) transforms covariantly, $\delta\boldsymbol{\chi} = g\boldsymbol{\chi}\times(\boldsymbol{\omega}_\parallel + \boldsymbol{\omega}_\perp) = g\boldsymbol{\chi}\times\boldsymbol{\omega}$. Here the local rotation of $\boldsymbol{n}$, $\delta\boldsymbol{n}(x) = g\boldsymbol{n}(x)\times\boldsymbol{\theta}_\perp(x)$, leads to $\delta\boldsymbol{\chi} = 0$ on $\boldsymbol{\chi} = 0$. Here the $\mathrm{U}(1)^\omega_{\mathrm{local}}$ part in $SU(2)^\omega_{\mathrm{local}}$ is not affected by this condition. Hence, the gauge symmetry corresponding to $\boldsymbol{\omega}_\parallel(x)$ remains unbroken.

Therefore, if we impose the condition (10) to the CFN–Yang–Mills theory, we have a gauge theory (Yang–Mills theory II) with the local gauge symmetry $SU(2)^{\omega=\theta}_{\mathrm{local}}$ corresponding to the gauge transformation parameter $\boldsymbol{\omega}(x) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x))$, which is a diagonal $SU(2)$ part of the original $SU(2)^\omega_{\mathrm{local}}\times[SU(2)/U(1)]^\theta_{\mathrm{local}}$. The local gauge symmetry $SU(2)^{\omega=\theta}_{\mathrm{local}}$ of the Yang–Mills theory II is the same as gauge symmetry II.

3 BRST symmetry and Faddeev–Popov ghost term

According to the clarification of the symmetry in the CFN–Yang–Mills theory explained above, we can obtain the unique Faddeev–Popov ghost term associated to the gauge fixing adopted in quantization. This is an advantage of our interpretation of the CFN–Yang–Mills theory.

3.1 Determination of BRST transformations

We introduce two kinds of ghosts, $\mathbb{C}_\omega$ and $\mathbb{C}_\theta$ corresponding to $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$, respectively. Then, by requiring

$$\boldsymbol{n}\cdot\mathbb{C}_\theta = 0, \quad (14)$$

the BRST transformations for A_μ and $\boldsymbol{n}$ are determined as follows:

$$\boldsymbol{\delta}_{\mathrm{B}}\mathrm{A}_\mu = D_\mu[\mathrm{A}]\mathbb{C}_\omega, \quad \boldsymbol{\delta}_{\mathrm{B}}\boldsymbol{n} = g\boldsymbol{n}\times\mathbb{C}_\theta. \quad (15)$$

3.1.1 ω sector

By imposing the nilpotency for A_μ, i.e., $\boldsymbol{\delta}_{\mathrm{B}}^2\mathrm{A}_\mu \equiv 0$, we can determine the BRST transformation for $\mathbb{C}_\omega$:

$$\boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\omega = -\frac{g}{2}\mathbb{C}_\omega\times\mathbb{C}_\omega. \quad (16)$$

as in the usual case. The nilpotency for $\mathbb{C}_\omega$, i.e., $\boldsymbol{\delta}_{\mathrm{B}}^2\mathbb{C}_\omega \equiv 0$ can be checked in the same way as in the usual case.

In a way similar to the ordinary case, we can introduce the antighost $\bar{\mathbb{C}}_\omega$ and the Nakanishi–Lautrup (NL) auxiliary field $\mathbb{N}_\omega$ obeying the BRST transformations:

$$\boldsymbol{\delta}_{\mathrm{B}}\bar{\mathbb{C}}_\omega = \mathrm{i}\mathbb{N}_\omega, \quad \boldsymbol{\delta}_{\mathrm{B}}\mathbb{N}_\omega = 0. \quad (17)$$

The nilpotency for $\bar{\mathbb{C}}_\omega$ and $\mathbb{N}_\omega$ is trivial, i.e., $\boldsymbol{\delta}_{\mathrm{B}}^2\bar{\mathbb{C}}_\omega \equiv 0$ and $\boldsymbol{\delta}_{\mathrm{B}}^2\mathbb{N}_\omega \equiv 0$.

3.1.2 θ sector

First, imposing the nilpotency for $\boldsymbol{n}$ yields the relation

$$0 \equiv \boldsymbol{\delta}_{\mathrm{B}}^2\boldsymbol{n} = g\boldsymbol{n}\times\boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\theta, \quad (18)$$

where we have used $\boldsymbol{n}\cdot\mathbb{C}_\theta = 0$ and $\mathbb{C}_\theta\cdot\mathbb{C}_\theta = 0$. Second, requiring $\boldsymbol{\delta}_{\mathrm{B}}(\boldsymbol{n}\cdot\mathbb{C}_\theta) = 0$, i.e.,

$$0 \equiv \boldsymbol{\delta}_{\mathrm{B}}(\boldsymbol{n}\cdot\mathbb{C}_\theta) = \boldsymbol{n}\cdot(g\mathbb{C}_\theta\times\mathbb{C}_\theta + \boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\theta), \quad (19)$$

leads to another relation,

$$\boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\theta = -g\mathbb{C}_\theta\times\mathbb{C}_\theta + \boldsymbol{f}_\perp, \quad (20)$$

where $\boldsymbol{f}_\perp$ denotes an arbitrary function perpendicular to $\boldsymbol{n}$. Substituting the relation (20) into (18), we have

$$\boldsymbol{n}\times\boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\theta = \boldsymbol{n}\times\boldsymbol{f}_\perp = 0, \quad (21)$$

implying $\boldsymbol{f}_\perp = 0$. Hence the BRST transformation for $\mathbb{C}_\theta$ is determined as

$$\boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\theta = -g\mathbb{C}_\theta\times\mathbb{C}_\theta, \quad (22)$$

where $\mathbb{C}_\theta\times\mathbb{C}_\theta$ is parallel to $\boldsymbol{n}$ due to (18) and (22).[2]The nilpotency $\boldsymbol{\delta}_{\mathrm{B}}^2\mathbb{C}_\theta \equiv 0$ can be checked without difficulty.

We introduce the antighost $\bar{\mathbb{C}}_\theta$ and the NL field N_θ such that they have the BRST transformations

$$\boldsymbol{\delta}_{\mathrm{B}}\bar{\mathbb{C}}_\theta = \mathrm{i}\mathbb{N}_\theta, \quad \boldsymbol{\delta}_{\mathrm{B}}\mathbb{N}_\theta = 0. \quad (23)$$

[2] Alternatively, (22) is shown as follows. The decomposition of the BRST transformation of $\mathbb{C}_\theta$ into the parallel and perpendicular parts to $\boldsymbol{n}$ yields

$$\boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\theta = (\boldsymbol{n}\cdot\boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\theta)\boldsymbol{n} + (\boldsymbol{n}\times\boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\theta)\times\boldsymbol{n} = -g\{\boldsymbol{n}\cdot(\mathbb{C}_\theta\times\mathbb{C}_\theta)\}\boldsymbol{n}$$
$$= -g\mathbb{C}_\theta\times\mathbb{C}_\theta,$$

where we have used (18) and (19), i.e., $\boldsymbol{n}\cdot\boldsymbol{\delta}_{\mathrm{B}}\mathbb{C}_\theta = -g\boldsymbol{n}\cdot(\mathbb{C}_\theta\times\mathbb{C}_\theta)$, in the second equality, and we have used the fact that $\mathbb{C}_\theta\times\mathbb{C}_\theta$ is parallel to $\boldsymbol{n}$ in the last step.

The nilpotency for $\bar{\mathbb{C}}_\theta$ and $\mathbb{N}_\theta$ is trivial, i.e., $\boldsymbol{\delta}_{\rm B}^2\bar{\mathbb{C}}_\theta \equiv 0$ and $\boldsymbol{\delta}_{\rm B}^2\mathbb{N}_\theta \equiv 0$.

We have not yet imposed any conditions on $\bar{\mathbb{C}}_\theta$, $\mathbb{N}_\theta$, although we imposed $\boldsymbol{n}\cdot\mathbb{C}_\theta = 0$ on $\mathbb{C}_\theta$. Note that $\bar{\mathbb{C}}_\theta$ and $\mathbb{N}_\theta$ are not necessarily perpendicular to $\boldsymbol{n}$. In light of the fact that $\mathbb{C}_\theta$ has two degrees of freedom, we impose

$$\boldsymbol{n}\cdot\bar{\mathbb{C}}_\theta = 0 \tag{24}$$

and $\boldsymbol{\delta}_{\rm B}(\boldsymbol{n}\cdot\bar{\mathbb{C}}_\theta) \equiv 0$. Then we find

$$\begin{aligned} 0 \equiv \boldsymbol{\delta}_{\rm B}(\boldsymbol{n}\cdot\bar{\mathbb{C}}_\theta) &= \boldsymbol{n}\cdot(g\mathbb{C}_\theta\times\bar{\mathbb{C}}_\theta + \boldsymbol{\delta}_{\rm B}\bar{\mathbb{C}}_\theta) \\ &= \boldsymbol{n}\cdot(g\mathbb{C}_\theta\times\bar{\mathbb{C}}_\theta + {\rm i}\mathbb{N}_\theta). \end{aligned} \tag{25}$$

This condition implies that the parallel component $\mathbb{N}_\theta^{\|}$ is non-zero and written in terms of $\mathbb{C}_\theta$, $\bar{\mathbb{C}}_\theta$ and $\boldsymbol{n}$ as follows:

$$\begin{aligned} \mathbb{N}_\theta^{\|} := \boldsymbol{n}(\boldsymbol{n}\cdot\mathbb{N}_\theta) &= {\rm i}g\boldsymbol{n}[\boldsymbol{n}\cdot(\mathbb{C}_\theta\times\bar{\mathbb{C}}_\theta)] \\ &= {\rm i}g(\mathbb{C}_\theta\times\bar{\mathbb{C}}_\theta), \end{aligned} \tag{26}$$

where we have used the fact that $\mathbb{C}_\theta\times\bar{\mathbb{C}}_\theta$ is parallel to $\boldsymbol{n}$, since $\boldsymbol{n}\cdot\mathbb{C}_\theta = 0 = \boldsymbol{n}\cdot\bar{\mathbb{C}}_\theta$. Therefore, $\mathbb{N}_\theta^{\|}$ is not the independent degree of freedom.

3.1.3 Summarizing BRST transformations

Thus we have determined the nilpotent BRST transformations for the CFN variables based on a new interpretation [14] of the CFN–Yang–Mills theory.

The CFN variables obey the BRST transformations:

$$\boldsymbol{\delta}_{\rm B}\boldsymbol{n} = g\boldsymbol{n}\times\mathbb{C}_\theta^{\perp}, \tag{27a}$$

$$\boldsymbol{\delta}_{\rm B}c_\mu = g(\boldsymbol{n}\times\mathbb{A}_\mu)\cdot(\mathbb{C}_\omega^{\perp}-\mathbb{C}_\theta^{\perp}) + \boldsymbol{n}\cdot\partial_\mu\mathbb{C}_\omega, \tag{27b}$$

$$\boldsymbol{\delta}_{\rm B}\mathbb{X}_\mu = g\mathbb{X}_\mu\times(\mathbb{C}_\omega^{\|}+\mathbb{C}_\theta^{\perp}) + D_\mu[\mathbb{V}](\mathbb{C}_\omega^{\perp}-\mathbb{C}_\theta^{\perp}), \tag{27c}$$

which are supplemented by the BRST transformations in the ω sector

$$\boldsymbol{\delta}_{\rm B}\mathbb{C}_\omega = -\frac{g}{2}\mathbb{C}_\omega\times\mathbb{C}_\omega, \tag{27d}$$

$$\boldsymbol{\delta}_{\rm B}\bar{\mathbb{C}}_\omega = {\rm i}\mathbb{N}_\omega, \tag{27e}$$

$$\boldsymbol{\delta}_{\rm B}\mathbb{N}_\omega = 0, \tag{27f}$$

and the BRST transformations in the θ sector

$$\boldsymbol{\delta}_{\rm B}\mathbb{C}_\theta^{\perp} = -g\mathbb{C}_\theta^{\perp}\times\mathbb{C}_\theta^{\perp}, \tag{27g}$$

$$\boldsymbol{\delta}_{\rm B}\bar{\mathbb{C}}_\theta^{\perp} = {\rm i}\mathbb{N}_\theta^{\perp} - g\mathbb{C}_\theta^{\perp}\times\bar{\mathbb{C}}_\theta^{\perp}, \tag{27h}$$

$$\begin{aligned} \boldsymbol{\delta}_{\rm B}\mathbb{N}_\theta^{\perp} &= g\mathbb{N}_\theta^{\perp}\times\mathbb{C}_\theta - g^2{\rm i}(\mathbb{C}_\theta\cdot\bar{\mathbb{C}}_\theta)\mathbb{C}_\theta \\ &= g(\mathbb{N}_\theta^{\perp} - {\rm i}g\mathbb{C}_\theta\times\bar{\mathbb{C}}_\theta)\times\mathbb{C}_\theta \\ &= -\boldsymbol{\delta}_{\rm B}\mathbb{N}_\theta^{\|}, \end{aligned} \tag{27i}$$

where we have explicitly written the BRST transformation of the perpendicular $\mathbb{N}_\theta^{\perp}$ component in conjunction to the parallel $\mathbb{N}_\theta^{\|}$ component.[3] Here $\mathbb{C}_\theta^{\perp}$, $\bar{\mathbb{C}}_\theta^{\perp}$ and $\mathbb{N}_\theta^{\perp}$ have two independent degrees of freedom perpendicular to $\boldsymbol{n}$.

3.2 Gauge-fixing and FP ghost term

The gauge-fixing (GF) and the associated Faddeev–Popov (FP) ghost term is written as follows:

$$\mathcal{L}_{\rm GF+FP} = \mathcal{L}_{\rm GF+FP}^{\omega} + \mathcal{L}_{\rm GF+FP}^{\theta}, \tag{28a}$$

$$\mathcal{L}_{\rm GF+FP}^{\omega} := -{\rm i}\boldsymbol{\delta}_{\rm B}(\bar{\mathbb{C}}_\omega\cdot\mathbb{F}_\omega) = -{\rm i}\boldsymbol{\delta}_{\rm B}(\bar{\mathbb{C}}_\omega\cdot\partial^\mu\mathbb{A}_\mu), \tag{28b}$$

$$\mathcal{L}_{\rm GF+FP}^{\theta} := -{\rm i}\boldsymbol{\delta}_{\rm B}(\bar{\mathbb{C}}_\theta\cdot\mathbb{F}_\theta) = -{\rm i}\boldsymbol{\delta}_{\rm B}(\bar{\mathbb{C}}_\theta\cdot D^\mu[\mathbb{V}]\mathbb{X}_\mu). \tag{28c}$$

The first term $\mathcal{L}_{\rm GF+FP}^{\omega}$ is calculated in a way similar to the ordinary Lorentz gauge as follows:

$$\mathcal{L}_{\rm GF+FP}^{\omega} = \mathbb{N}_\omega\cdot\partial^\mu\mathbb{A}_\mu + {\rm i}\bar{\mathbb{C}}_\omega\cdot\partial^\mu D_\mu[\mathbb{A}]\mathbb{C}_\omega. \tag{29}$$

In calculating the second term $\mathcal{L}_{\rm GF+FP}^{\theta}$, a useful observation is that the new MAG condition $\mathbb{F}_\theta = D^\mu[\mathbb{V}]\mathbb{X}_\mu$ can be rewritten in terms of $\mathbb{A}_\mu$ and $\boldsymbol{n}$ as

$$\begin{aligned} \mathbb{F}_\theta = D^\mu[\mathbb{V}]\mathbb{X}_\mu &= D^\mu[\mathbb{A}]\mathbb{X}_\mu \\ &= g^{-1}D^\mu[\mathbb{A}](\boldsymbol{n}\times D_\mu[\mathbb{A}]\boldsymbol{n}) \\ &= g^{-1}\boldsymbol{n}\times D^\mu[\mathbb{A}]D_\mu[\mathbb{A}]\boldsymbol{n}. \end{aligned} \tag{30}$$

Then, it is straightforward but a little bit tedious to show that

$$\begin{aligned} \mathcal{L}_{\rm GF+FP}^{\theta} = {} & \mathbb{N}_\theta^{\perp}\cdot(D^\mu[\mathbb{V}]\mathbb{X}_\mu) \\ & -{\rm i}\bar{\mathbb{C}}_\theta\cdot D^\mu[\mathbb{V}-\mathbb{X}]D_\mu[\mathbb{V}+\mathbb{X}](\mathbb{C}_\theta-\mathbb{C}_\omega), \end{aligned} \tag{31}$$

where we have used that $\bar{\mathbb{C}}_\theta\cdot\boldsymbol{n} = 0 = \mathbb{C}_\theta\cdot\boldsymbol{n}$ and $\boldsymbol{n}\cdot D_\mu[\mathbb{A}]\boldsymbol{n} = 0$. This is one of the main results of this paper.

Note that $\mathcal{L}_{\rm GF+FP}^{\theta}$ includes the ghost field $\mathbb{C}_\omega$ which cannot be eliminated from (31) by shifting the variable: $\mathbb{C}_\theta \to \mathbb{C}_\theta - \mathbb{C}_\omega$. This is because $\mathbb{C}_\omega$ has three degrees of freedom, while $\mathbb{C}_\theta$ has two degrees of freedom.

In the one-loop calculation, however, we can eliminate $\mathbb{C}_\omega$ by shifting $\mathbb{C}_\theta \to \mathbb{C}_\theta - \mathbb{C}_\omega^{\perp}$, if we treat $\mathbb{V}$ as a background and $\mathbb{X}$, $\mathbb{C}$, $\bar{\mathbb{C}}$ as the quantum fluctuation around it:

$$\begin{aligned} &\mathcal{L}_{\rm GF+FP}^{\theta} \\ &= \mathbb{N}_\theta^{\perp}\cdot(D^\mu[\mathbb{V}]\mathbb{X}_\mu) - {\rm i}\bar{\mathbb{C}}_\theta\cdot D^\mu[\mathbb{V}]D_\mu[\mathbb{V}](\mathbb{C}_\theta-\mathbb{C}_\omega) \\ &\quad + \dots \\ &= \mathbb{N}_\theta^{\perp}\cdot(D^\mu[\mathbb{V}]\mathbb{X}_\mu) - {\rm i}\bar{\mathbb{C}}_\theta\cdot D^\mu[\mathbb{V}]D_\mu[\mathbb{V}](\mathbb{C}_\theta-\mathbb{C}_\omega^{\perp}) \end{aligned}$$

[3] Equation (27i) is obtained from the nilpotency for $\bar{\mathbb{C}}_\theta$:

$$\begin{aligned} 0 \equiv \boldsymbol{\delta}_{\rm B}^2\bar{\mathbb{C}}_\theta = {\rm i}\boldsymbol{\delta}_{\rm B}\mathbb{N}_\theta &= {\rm i}\boldsymbol{\delta}_{\rm B}(\mathbb{N}_\theta^{\perp}+\mathbb{N}_\theta^{\|}) \\ &= {\rm i}\boldsymbol{\delta}_{\rm B}(\mathbb{N}_\theta^{\perp}+{\rm i}g\mathbb{C}_\theta\times\bar{\mathbb{C}}_\theta) \\ &= {\rm i}\boldsymbol{\delta}_{\rm B}\mathbb{N}_\theta^{\perp} + {\rm i}g\mathbb{C}_\theta\times\mathbb{N}_\theta^{\perp} - g^2(\mathbb{C}_\theta\cdot\bar{\mathbb{C}}_\theta)\mathbb{C}_\theta. \end{aligned}$$

$$+\dots$$
$$= \mathbb{N}_\theta^\perp \cdot (D^\mu[\mathbb{V}]\mathbb{X}_\mu) - \mathrm{i}\bar{\mathbb{C}}_\theta \cdot D^\mu[\mathbb{V}]D_\mu[\mathbb{V}]\mathbb{C}_\theta + \dots \quad (32)$$

Thus, the previous result in the one-loop level [12] is not affected by the correct treatment of the FP ghost term.

Moreover, observing $\boldsymbol{\delta}_\mathrm{B}(\mathbb{N}_\omega\cdot\mathbb{N}_\omega) = 0$, $\boldsymbol{\delta}_\mathrm{B}(\mathbb{N}_\theta\cdot\mathbb{N}_\theta) = 0$, and $\boldsymbol{\delta}_\mathrm{B}(\mathbb{N}_\theta+\zeta\mathbb{N}_\omega)^2 = 0$, we are allowed to add the following term to the GF + FP term:

$$\mathcal{L}_N := \frac{\alpha}{2}(\mathbb{N}_\omega + \zeta\mathbb{N}_\theta)\cdot(\mathbb{N}_\omega + \zeta\mathbb{N}_\theta) + \frac{\beta}{2}\mathbb{N}_\omega\cdot\mathbb{N}_\omega. \quad (33)$$

The usefulness of this term is demonstrated in the modified new MAG in the final part of this paper.

4 Primed gauge and BRST transformations

4.1 Primed gauge transformations

Another way to describe the gauge transformation property of the CFN–Yang–Mills theory in terms of the CFN variables $(\boldsymbol{n}, c_\mu, \mathbb{X}_\mu)$ is to introduce

$$\boldsymbol{\omega}' := \boldsymbol{\omega}_\parallel + \boldsymbol{\theta}_\perp, \quad \boldsymbol{\theta}' := \boldsymbol{\omega}_\perp - \boldsymbol{\theta}_\perp, \quad (\boldsymbol{\theta}'\cdot\boldsymbol{n} = 0), \quad (34)$$

rather than $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$. The relation between the two sets of gauge transformation parameters,

$$\boldsymbol{\omega}' + \boldsymbol{\theta}' = \boldsymbol{\omega}_\parallel + \boldsymbol{\omega}_\perp = \boldsymbol{\omega}, \quad (35a)$$
$$\boldsymbol{n}\times(\boldsymbol{\omega}'\times\boldsymbol{n}) = \boldsymbol{\omega}'_\perp = \boldsymbol{\theta}_\perp = \boldsymbol{\theta}, \quad (35b)$$

yields another view of the local gauge transformations:

$$\delta \mathrm{A}_\mu = D_\mu[\mathrm{A}](\boldsymbol{\omega}' + \boldsymbol{\theta}'), \quad (36a)$$
$$\delta\boldsymbol{n} = g\boldsymbol{n}\times\boldsymbol{\omega}'_\perp = g\boldsymbol{n}\times\boldsymbol{\omega}', \quad (36b)$$
$$\delta c_\mu = g(\mathrm{A}_\mu\times\boldsymbol{n})\cdot\boldsymbol{\theta}' + \boldsymbol{n}\cdot\partial_\mu(\boldsymbol{\omega}'+\boldsymbol{\theta}'), \quad (36c)$$
$$\delta\mathbb{X}_\mu = g\mathbb{X}_\mu\times\boldsymbol{\omega}' + D_\mu[\mathbb{V}]\boldsymbol{\theta}'. \quad (36d)$$

In fact, $\boldsymbol{\theta}' = 0$, i.e., $\boldsymbol{\omega}_\perp = \boldsymbol{\theta}_\perp$ reproduces the local gauge transformation II.

4.2 Primed BRST transformations

By introducing the ghost fields $\mathbb{C}'_\omega$ and $\mathbb{C}'_\theta$ corresponding to $\boldsymbol{\omega}'$ and $\boldsymbol{\theta}'$ respectively, we can introduce the BRST transformations of the CFN variables $(\boldsymbol{n}, c_\mu, \mathbb{X}_\mu)$ in addition to the original Yang–Mills field A_μ. We have

$$\boldsymbol{\delta}_\mathrm{B}\mathrm{A}_\mu = D_\mu[\mathrm{A}](\mathbb{C}'_\omega + \mathbb{C}'_\theta), \quad (37a)$$
$$\boldsymbol{\delta}_\mathrm{B}\boldsymbol{n} = g\boldsymbol{n}\times\mathbb{C}'_\omega, \quad (37b)$$
$$\boldsymbol{\delta}_\mathrm{B}c_\mu = g(\mathrm{A}_\mu\times\boldsymbol{n})\cdot\mathbb{C}'_\theta + \boldsymbol{n}\cdot\partial_\mu(\mathbb{C}'_\omega+\mathbb{C}'_\theta), \quad (37c)$$
$$\boldsymbol{\delta}_\mathrm{B}\mathbb{X}_\mu = g\mathbb{X}_\mu\times\mathbb{C}'_\omega + D_\mu[\mathbb{V}]\mathbb{C}'_\theta. \quad (37d)$$

In a way similar to the above, the BRST transformations of the ghost fields, $\mathbb{C}'_\omega$, $\mathbb{C}'_\theta$, and antighost fields, $\bar{\mathbb{C}}'_\omega$, $\bar{\mathbb{C}}'_\theta$, can be determined as well as the NL fields $\mathbb{N}'_\theta$, $\mathbb{N}'_\omega$:

$$\boldsymbol{\delta}_\mathrm{B}\mathbb{C}'_\theta = -g\mathbb{C}'_\omega\times\mathbb{C}'_\theta, \quad (37e)$$
$$\boldsymbol{\delta}_\mathrm{B}\bar{\mathbb{C}}'_\theta = \mathrm{i}\mathbb{N}'_\theta, \quad (37f)$$
$$\boldsymbol{\delta}_\mathrm{B}\mathbb{N}'_\theta = 0, \quad (37g)$$
$$\boldsymbol{\delta}_\mathrm{B}\mathbb{C}'_\omega = -\frac{g}{2}(\mathbb{C}'_\omega\times\mathbb{C}'_\omega + \mathbb{C}'_\theta\times\mathbb{C}'_\theta), \quad (37h)$$
$$\boldsymbol{\delta}_\mathrm{B}\bar{\mathbb{C}}'_\omega = \mathrm{i}\mathbb{N}'_\omega, \quad (37i)$$
$$\boldsymbol{\delta}_\mathrm{B}\mathbb{N}'_\omega = 0, \quad (37j)$$

where $\boldsymbol{n}\cdot\mathbb{C}'_\theta = 0 = \boldsymbol{n}\cdot\bar{\mathbb{C}}'_\theta$, $\boldsymbol{n}\cdot\mathbb{N}'_\theta = \mathrm{i}g\boldsymbol{n}\cdot(\mathbb{C}'_\theta\times\bar{\mathbb{C}}'_\theta)$.

4.3 Primed gauge fixing and FP ghost term

We impose two gauge fixing conditions: $\mathbb{F}'_\omega = \mathbb{F}_\omega = \partial^\mu\mathrm{A}_\mu$ for fixing the gauge degrees of freedom $\boldsymbol{\omega}'$ and $\mathbb{F}'_\theta = \mathbb{F}_\theta = D^\mu[\mathbb{V}]\mathbb{X}_\mu$ for fixing gauge degrees of freedom $\boldsymbol{\theta}'$. Then we can obtain the primed GF + FP term in a way similar to the one above,

$$\begin{aligned}
\mathcal{L}_{\mathrm{GF+FP}} &= \mathcal{L}'_\omega + \mathcal{L}'_\theta,\\
\mathcal{L}'_\omega &:= -\mathrm{i}\boldsymbol{\delta}_\mathrm{B}(\bar{\mathbb{C}}'_\omega\cdot\mathbb{F}_\omega) = -\mathrm{i}\boldsymbol{\delta}_\mathrm{B}(\bar{\mathbb{C}}'_\omega\cdot\partial_\mu\mathrm{A}_\mu) \quad (38)\\
&= \mathbb{N}'_\omega\cdot\partial_\mu\mathrm{A}_\mu + \mathrm{i}\bar{\mathbb{C}}'_\omega\cdot\partial^\mu D_\mu[\mathrm{A}](\mathbb{C}'_\omega+\mathbb{C}'_\theta),\\
\mathcal{L}'_\theta &:= -\mathrm{i}\boldsymbol{\delta}_\mathrm{B}(\bar{\mathbb{C}}'_\theta\cdot\mathbb{F}_\theta) = -\mathrm{i}\boldsymbol{\delta}_\mathrm{B}(\bar{\mathbb{C}}'_\theta\cdot D_\mu[\mathbb{V}]\mathbb{X}_\mu)\\
&= \mathbb{N}'_\theta\cdot D_\mu[\mathbb{V}]\mathbb{X}_\mu\\
&\quad +\mathrm{i}\bar{\mathbb{C}}'_\theta\cdot\{g(D_\mu[\mathbb{V}]\mathbb{X}^\mu)\times(\mathbb{C}'_\omega-\mathbb{C}'_\theta)\\
&\quad +D_\mu[\mathbb{V}-\mathbb{X}]D_\mu[\mathbb{V}+\mathbb{X}]\mathbb{C}'_\theta\}. \quad (39)
\end{aligned}$$

The second term is further rewritten as

$$\begin{aligned}
\mathcal{L}'_\theta &= \mathbb{N}'_\theta\cdot D_\mu[\mathbb{V}]\mathbb{X}_\mu\\
&\quad +\mathrm{i}\bar{\mathbb{C}}'_\theta\cdot\{-g(\mathbb{C}'_\omega)_\parallel\times(D_\mu[\mathbb{V}]\mathbb{X}^\mu)\\
&\quad +D_\mu[\mathbb{V}-\mathbb{X}]D_\mu[\mathbb{V}+\mathbb{X}]\mathbb{C}'_\theta\}\\
&= \mathbb{N}'_\theta\cdot D_\mu[\mathbb{V}]\mathbb{X}_\mu \quad (40)\\
&\quad +\mathrm{i}\bar{\mathbb{C}}'_\theta\cdot D_\mu[\mathbb{V}-\mathbb{X}]D_\mu[\mathbb{V}+\mathbb{X}]\{\mathbb{C}'_\theta+(\mathbb{C}'_\omega)_\parallel\}.
\end{aligned}$$

Note that (38) and (40) agree with (29) and (31) under the identification

$$\mathbb{C}_\omega = \mathbb{C}'_\omega + \mathbb{C}'_\theta, \quad \mathbb{C}_\theta = (\mathbb{C}'_\omega)_\perp. \quad (41)$$

This is expected from the observation of the correspondence between the original parameters and ghosts $\boldsymbol{\omega}\to\mathbb{C}_\omega$, $\boldsymbol{\theta}\to\mathbb{C}_\theta$, and the primed parameters and ghosts $\boldsymbol{\omega}'\to\mathbb{C}'_\omega$, $\boldsymbol{\theta}'\to\mathbb{C}'_\theta$. This can be a cross-check for the correctness of our calculations.

5 Anti-BRST transformation and its application

By the formal replacement $\mathbb{C} \to \bar{\mathbb{C}}$, $\bar{\mathbb{C}} \to \mathbb{C}$ and $\mathbb{N} \to \bar{\mathbb{N}}$, we begin to determine the anti-BRST transformation.

5.1 Anti-BRST transformation

The anti-BRST transformations for ω sector are obtained as

$$\bar{\boldsymbol{\delta}}_{\rm B}\mathrm{A}_\mu = D_\mu[\mathrm{A}]\bar{\mathbb{C}}_\omega, \tag{42a}$$

$$\bar{\boldsymbol{\delta}}_{\rm B}\bar{\mathbb{C}}_\omega = -\frac{g}{2}\bar{\mathbb{C}}_\omega \times \bar{\mathbb{C}}_\omega, \tag{42b}$$

$$\bar{\boldsymbol{\delta}}_{\rm B}\mathbb{C}_\omega = \mathrm{i}\bar{\mathbb{N}}_\omega, \tag{42c}$$

$$\bar{\boldsymbol{\delta}}_{\rm B}\bar{\mathbb{N}}_\omega = 0. \tag{42d}$$

We require $\{\boldsymbol{\delta}_{\rm B}, \bar{\boldsymbol{\delta}}_{\rm B}\} = 0$ to obtain the relationship between $\mathbb{N}$ and $\bar{\mathbb{N}}$. For A_μ, using $\boldsymbol{\delta}_{\rm B}\bar{\boldsymbol{\delta}}_{\rm B}\mathrm{A}_\mu = \boldsymbol{\delta}_{\rm B}(D_\mu[\mathrm{A}]\bar{\mathbb{C}}_\omega) = \mathrm{i}D_\mu[\mathrm{A}]\mathbb{N}_\omega + gD_\mu[\mathrm{A}]\mathbb{C}_\omega \times \bar{\mathbb{C}}_\omega$, we obtain

$$\begin{aligned} 0 &\equiv (\boldsymbol{\delta}_{\rm B}\bar{\boldsymbol{\delta}}_{\rm B} + \bar{\boldsymbol{\delta}}_{\rm B}\boldsymbol{\delta}_{\rm B})\mathrm{A}_\mu \\ &= \mathrm{i}D_\mu[\mathrm{A}](\mathbb{N}_\omega + \bar{\mathbb{N}}_\omega - \mathrm{i}g\mathbb{C}_\omega \times \bar{\mathbb{C}}_\omega), \end{aligned} \tag{43}$$

yielding the relation

$$\mathbb{N}_\omega + \bar{\mathbb{N}}_\omega = \mathrm{i}g\mathbb{C}_\omega \times \bar{\mathbb{C}}_\omega. \tag{44}$$

On the other hand, the anti-BRST transformations for the θ sector are found to be

$$\bar{\boldsymbol{\delta}}_{\rm B}\boldsymbol{n} = g\boldsymbol{n} \times \bar{\mathbb{C}}_\theta^\perp, \tag{45a}$$

$$\bar{\boldsymbol{\delta}}_{\rm B}\bar{\mathbb{C}}_\theta^\perp = -g\bar{\mathbb{C}}_\theta^\perp \times \bar{\mathbb{C}}_\theta^\perp, \tag{45b}$$

$$\bar{\boldsymbol{\delta}}_{\rm B}\mathbb{C}_\theta^\perp = \mathrm{i}\bar{\mathbb{N}}_\theta^\perp, \tag{45c}$$

$$\bar{\boldsymbol{\delta}}_{\rm B}\bar{\mathbb{N}}_\theta^\perp = 0, \tag{45d}$$

where $\bar{\mathbb{N}}_\theta$ have two independent degrees of freedom $\bar{\mathbb{N}}_\theta^\perp$, since

$$\mathbb{N}_\theta^\parallel = \mathrm{i}g\boldsymbol{n}[\boldsymbol{n} \cdot (\bar{\mathbb{C}}_\theta^\perp \times \mathbb{C}_\theta^\perp)] = \mathrm{i}g\bar{\mathbb{C}}_\theta^\perp \times \mathbb{C}_\theta^\perp. \tag{46}$$

For $\boldsymbol{n}$, we calculate $\boldsymbol{\delta}_{\rm B}\bar{\boldsymbol{\delta}}_{\rm B}\boldsymbol{n} = \boldsymbol{\delta}_{\rm B}(g\boldsymbol{n} \times \bar{\mathbb{C}}_\theta^\perp) = -g^2\boldsymbol{n}(\mathbb{C}_\theta^\perp \cdot \bar{\mathbb{C}}_\theta^\perp) + \mathrm{i}g\boldsymbol{n} \times \mathbb{N}_\theta$, and hence

$$0 \equiv (\boldsymbol{\delta}_{\rm B}\bar{\boldsymbol{\delta}}_{\rm B} + \bar{\boldsymbol{\delta}}_{\rm B}\boldsymbol{\delta}_{\rm B})\boldsymbol{n} = \mathrm{i}g\boldsymbol{n} \times (\mathbb{N}_\theta + \bar{\mathbb{N}}_\theta), \tag{47}$$

which implies the relationship

$$\mathbb{N}_\theta^\perp + \bar{\mathbb{N}}_\theta^\perp = 0, \quad \mathbb{N}_\theta + \bar{\mathbb{N}}_\theta = \mathbb{N}_\theta^\parallel + \bar{\mathbb{N}}_\theta^\parallel = 2\mathrm{i}g\mathbb{C}_\theta^\perp \times \bar{\mathbb{C}}_\theta^\perp. \tag{48}$$

We have checked that the requirement $\boldsymbol{\delta}_{\rm B}\bar{\boldsymbol{\delta}}_{\rm B} + \bar{\boldsymbol{\delta}}_{\rm B}\boldsymbol{\delta}_{\rm B} \equiv 0$ leads to no new relations among the fields.

5.2 Modified new MAG

As an application of the anti-BRST transformation, we propose a modified version [17] of the new MAG:

$$\begin{aligned} &S_{\rm GF+FP} \\ &= \int \mathrm{d}^D x \mathrm{i}\boldsymbol{\delta}_{\rm B}\bar{\boldsymbol{\delta}}_{\rm B}\left(\tfrac{1}{2}\mathbb{X}_\mu \cdot \mathbb{X}^\mu\right) \\ &= \int \mathrm{d}^D x\, \mathrm{i}^{-1}\boldsymbol{\delta}_{\rm B}[(\bar{\mathbb{C}}_\omega - \bar{\mathbb{C}}_\theta^\perp) \cdot D^\mu[\mathbb{V}]\mathbb{X}_\mu] \\ &= \int \mathrm{d}^D x \Big\{(\mathbb{N}_\omega^\perp - \mathbb{N}_\theta^\perp) \cdot (D^\mu[\mathbb{V}]\mathbb{X}_\mu) \\ &\quad -\mathrm{i}(\bar{\mathbb{C}}_\omega^\perp - \bar{\mathbb{C}}_\theta^\perp) \cdot D^\mu[\mathbb{V} - \mathbb{X}]D_\mu[\mathbb{V} + \mathbb{X}](\mathbb{C}_\theta^\perp - \mathbb{C}_\omega)\Big\}. \end{aligned} \tag{49}$$

This term is exact simultaneously in the BRST and anti-BRST transformation and invariant under the FP conjugation. This form of the FP term can be cast into a simplified form by taking an appropriate linear combination of two NL fields, $\mathbb{N}_\omega^\perp$ and $\mathbb{N}_\theta^\perp$, in (33) and by integrating out the NL fields.

6 Conclusion

In this paper we have determined the BRST and anti-BRST transformations of the CFN variables and those of the associated ghost, antighost and Nakanishi–Lautrup auxiliary fields. The general form of the gauge-fixing term for the new MAG and the correct form of the associated FP term are obtained explicitly based on the new interpretation [14] of the CFN–Yang–Mills theory. Although the general form is different from the previous one [12], it reduces in the one-loop level to the same form as that given in [12] and does not change the main result [12] obtained based on the one-loop expression. Finally, a modified form of the new MAG has been proposed using the BRST and anti-BRST transformations.

Acknowledgements. This work is supported by Grant-in-Aid for Scientific Research (C)14540243 from Japan Society for the Promotion of Science (JSPS), and in part by Grant-in-Aid for Scientific Research on Priority Areas (B)13135203 from the Ministry of Education, Culture, Sports, Science and Technology (MEXT).

References

1. Y.M. Cho, Phys. Rev. D **21**, 1080 (1980); D **23**, 2415 (1981); Phys. Rev. Lett. **46**, 302 (1981)
2. L. Faddeev, A.J. Niemi, Phys. Rev. Lett. **82**, 1624 (1999) [hep-th/9807069]
3. L. Faddeev, A.J. Niemi, Phys. Lett. B **525**, 195 (2002) [hep-th/0101078]
4. S.V. Shabanov, Phys. Lett. B **458**, 322 (1999) [hep-th/9903223]; B **463**, 263 (1999) [hep-th/9907182]
5. H. Gies, Phys. Rev. D **63**, 125023 (2001) [hep-th/0102026]

6. L. Faddeev, A.J. Niemi, Nature **387**, 58 (1997) [hep-th/9610193]
7. E. Langmann, A.J. Niemi, Phys. Lett. B **463**, 252 (1999) [hep-th/9905147]
8. W.S. Bae, Y.M. Cho, S.W. Kimm, Phys. Rev. D **65**, 025005 (2001) [hep-th/0105163]
9. G.K. Savvidy, Phys. Lett. B **71**, 133 (1977)
10. N.K. Nielsen, P. Olesen, Nucl. Phys. B **144**, 376 (1978)
11. Y.M. Cho, M.L. Walker, D.G. Pak, JHEP **05**, 073 (2004) [hep-th/0209208]; Y.M. Cho, hep-th/0301013; Y.M. Cho, D.G. Pak, Phys. Rev. D **65**, 074027 (2002) [hep-th/0201179]
12. K.-I. Kondo, Phys. Lett. B **600**, 287 (2004) [hep-th/0404252]; Int. J. Mod. Phys. A (2005), to appear [hep-th/0410024]
13. S. Kato, K.-I. Kondo, T. Murakami, A. Shibata, T. Shinohara, hep-ph/0504054
14. K.-I. Kondo, T. Murakami, T. Shinohara, hep-th/0504107
15. G. 't Hooft, Nucl. Phys. B **190** [FS3], 455 (1981)
16. A. Kronfeld, M. Laursen, G. Schierholz, U.-J. Wiese, Phys. Lett. B **198**, 516 (1987)
17. K.-I. Kondo, Phys. Rev. D **58**, 105019 (1998) [hep-th/9801024]